Reversible Term Rewriting

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1st International Conference on
Formal Structures for Computation and Deduction

June 23, 2016

Porto, Portugal

Reversible Computing

computation	forward	backward
irreversible	$S \to S'$	✗
reversible	$S \to S'$	$S' \to S$

Applications in

- bidirectional program transformation
- cellular automata
- quantum computing
- ...

Motivation

Standard term rewriting is reversible, but its inverse relation may be

- non-deterministic
- non-confluent
- non-terminating

which makes it useless in our context

Our proposal is a **reversible extension** of term rewriting

Reversible Term Rewriting

Term Rewriting: An example

$$
\begin{array}{rrcl}
\beta_1 : & \mathsf{add}(0, y) & \to & y \\
\beta_2 : & \mathsf{add}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{add}(x, y)) \\
\beta_3 : & \mathsf{fst}(x, y) & \to & x
\end{array}
$$

$$\mathsf{fst}(\mathsf{add}(\mathsf{s}(0), 0), 0)$$

A posible reduction is

$$\mathsf{fst}(\mathsf{add}(\mathsf{s}(0), 0), 0) \to \mathsf{fst}(\mathsf{s}(\mathsf{add}(0, 0)), 0) \to \mathsf{fst}(\mathsf{s}(0), 0) \to \mathsf{s}(0)$$

but a deterministic inverse computation is not possible

- overlapping rhs
- erased variables

Trace terms

Proposal: Add trace terms

$$\beta(p, \sigma)$$

A trace term contains information about

- the applied rule (β)
- the position of the reduced subterm (p)
- the bindings of the erased variables (σ)

this approach is known as a *Landauer's embedding*

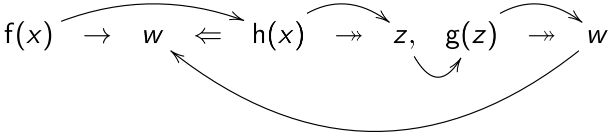
The computation becomes reversible

$$\begin{array}{rcl} \langle \mathsf{fst}(\mathsf{add}(\mathsf{s}(0),0),0),[\,]\rangle & \rightharpoonup_{\mathcal{R}} & \langle \mathsf{fst}(\mathsf{s}(\mathsf{add}(0,0)),0),[\beta_2(1,id)]\rangle \\ & \rightharpoonup_{\mathcal{R}} & \langle \mathsf{s}(\mathsf{add}(0,0)),[\beta_3(\epsilon,\{y \mapsto 0\}),\beta_2(1,id)]\rangle \\ & \rightharpoonup_{\mathcal{R}} & \langle \mathsf{s}(0),[\beta_1(1,id),\beta_3(\epsilon,\{y \mapsto 0\}),\beta_2(1,id)]\rangle \end{array}$$

DCTRSs

We consider **Deterministic Conditional TRSs** with

- **oriented:** $l \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_n \twoheadrightarrow t_n$
- **3-CTRS:** $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l) \cup \mathcal{V}ar(C)$
- **determinism:** $\mathcal{V}ar(s_i) \subseteq \mathcal{V}ar(l, \overline{t_{i-1}})$ for all $i = 1, \ldots, n$



Reversible extension

We extend standard term rewriting with relations

- $\rightharpoonup_{\mathcal{R}}$ forward
- $\leftharpoondown_{\mathcal{R}}$ backward
- $\rightleftharpoons_{\mathcal{R}}$ forward & backward

Reversible rewriting

Standard relation

$s \to_{\mathcal{R}} t$ iff there exist

- a position p in s
- a rewrite rule $\beta : l \to r \Leftarrow \overline{s_n \to t_n}$
- a substitution σ such that $s|_p = l\sigma$ and $s_i\sigma \to^*_{\mathcal{R}} t_i\sigma$ $\forall i = 1, \ldots, n$

Forward (reversible) relation

$\langle s, \pi \rangle \rightharpoonup_{\mathcal{R}} \langle t, \beta(p, \sigma', \pi_1, \ldots, \pi_n) : \pi \rangle$ iff there exist

- a position p in s
- a rewrite rule $\beta : l \to r \Leftarrow \overline{s_n \to t_n} \in \mathcal{R}$
- a substitution σ such that
  - $s|_p = l\sigma$
  - $\langle s_i\sigma, [\,] \rangle \rightharpoonup^*_{\mathcal{R}} \langle t_i\sigma, \pi_i \rangle$ $\forall i = 1, \ldots, n$, and $t = s[r\sigma]_p$
  - and $\sigma' = \sigma\restriction_{(\mathcal{V}ar(l) \setminus \mathcal{V}ar(r, \overline{s_n}, \overline{t_n})) \cup \bigcup_{i=1}^{n} \mathcal{V}ar(t_i) \setminus \mathcal{V}ar(r, \overline{s_{i+1,n}})}$

Reversible rewriting

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Reversible rewriting

Forward (reversible) relation

$$\langle s, \pi \rangle \rightharpoonup_{\mathcal{R}} \langle t, \beta(p, \sigma', \pi_1, \ldots, \pi_n) : \pi \rangle$$

Backward (reversible) relation

$$\langle t, \beta(p, \sigma', \pi_1, \ldots, \pi_n) : \pi \rangle \leftharpoondown_{\mathcal{R}} \langle s, \pi \rangle \text{ iff}$$

- $\beta : l \to r \Leftarrow \overline{s_n \twoheadrightarrow t_n} \in \mathcal{R}$
- there is a substitution θ such that
  - $\mathcal{D}om(\theta) = \mathcal{V}ar(r, \overline{s_n}) \backslash \mathcal{D}om(\sigma')$
  - $t|_p = r\theta$
  - $\langle t_i\, \theta \cup \sigma', \pi_i \rangle \leftharpoondown^*_{\mathcal{R}} \langle s_i\, \theta \cup \sigma', [\,] \rangle\ \forall i = 1, \ldots, n$
  - $s = t[l\, \theta \cup \sigma']_p$

Bindings: An example

$$\begin{array}{llcll} \beta_1: & \mathsf{f}(x,y,m) & \to & \mathsf{s}(w) & \Leftarrow \mathsf{h}(x) \twoheadrightarrow x, \mathsf{g}(y,4) \twoheadrightarrow w \\ \beta_2: & \mathsf{h}(0) & \to & 0 & \\ \beta_3: & \mathsf{h}(1) & \to & 1 & \\ \beta_4: & \mathsf{g}(x,y) & \to & x & \end{array}$$

$$\langle \mathsf{f}(0,2,4),[\,]\rangle \rightharpoonup_{\mathcal{R}} \langle \mathsf{s}(2),[\beta_1(\epsilon,\sigma',\pi_1,\pi_2)]\rangle$$

Here, $\sigma' = \{m \mapsto 4, x \mapsto 0\}$

- m is required to recover the value of the erased variable m
- x is required to perform the subderivation $\langle x,\pi_1\rangle \leftharpoondown_{\mathcal{R}} \langle \mathsf{h}(x),[\,]\rangle$ when applying a deterministic backward step

Results

Theorem (conservative)

Given terms s, t, if $s \to^*_{\mathcal{R}} t$, then for any trace π there exists a trace π' such that $\langle s, \pi \rangle \rightharpoonup^*_{\mathcal{R}} \langle t, \pi' \rangle$.

Theorem (reversibility)

$\langle s, \pi \rangle \rightharpoonup^* \langle t, \pi' \rangle$ *iff* $\langle t, \pi' \rangle \leftharpoondown^* \langle s, \pi \rangle$

- $\langle t, \pi' \rangle \leftharpoondown^* \langle s, \pi \rangle$ *is deterministic (thus confluent)*
- *and terminating*

Compiling reversibility

Compiling reversibility

We aim at *compiling* the reversible extension into the TRS rules

$$\mathcal{R} \nearrow \mathcal{R}_f \quad \mathcal{R} \searrow \mathcal{R}_b$$

$$\langle s, \pi \rangle \rightharpoonup_{\mathcal{R}} \langle t, \pi' \rangle \quad \text{iff} \quad s_\pi \rightarrow_{\mathcal{R}_f} t_{\pi'}$$

$$\langle t, \pi' \rangle \leftharpoondown_{\mathcal{R}} \langle s, \pi \rangle \quad \text{iff} \quad t_{\pi'} \rightarrow_{\mathcal{R}_b} s_\pi$$

Basic c-DCTRS

We consider **basic c-DCTRSs** where

- terms r and $\overline{t_n}$ are *constructor* terms
- terms $\overline{s_0}$ and $\overline{s_n}$ are *basic* terms (i.e., terms have the form $\mathsf{f}(c_1, \ldots, c_n)$, f is a defined function and $c_1, \ldots, c_n$ are constructor)

in these systems, reductions take place at topmost positions

$$\beta(p, \sigma, \overline{y}) \to \beta(\sigma, \overline{y})$$

we provide a flattening transformation from arbitrary systems to this form

Injectivization & Inversion

Injectivization: We replace each rule

$$\beta : \mathsf{f}(\overline{s_0}) \to r \Leftarrow \mathsf{f}_1(\overline{s_1}) \twoheadrightarrow t_1, \ldots, \mathsf{f}_n(\overline{s_n}) \twoheadrightarrow t_n$$

by a new rule of the form

$$\mathsf{f}^i(\overline{s_0}) \to \langle r, \beta(\overline{y}, \overline{w_n})\rangle \Leftarrow \mathsf{f}_1^i(\overline{s_1}) \twoheadrightarrow \langle t_1, w_1\rangle, \ldots, \mathsf{f}_n^i(\overline{s_n}) \twoheadrightarrow \langle t_n, w_n\rangle$$

where $\{\overline{y}\} = (\mathcal{V}ar(\overline{s_0})\backslash\mathcal{V}ar(r, \overline{s_n}, \overline{t_n})) \cup \bigcup_{i=1}^{n} \mathcal{V}ar(t_i)\backslash\mathcal{V}ar(r, \overline{s_{i+1,n}})$

Inversion: We replace each rule

$$\mathsf{f}^i(\overline{s_0}) \to \langle r, \beta(\overline{y}, \overline{w_n})\rangle \Leftarrow \mathsf{f}_1^i(\overline{s_1}) \twoheadrightarrow \langle t_1, w_1\rangle, \ldots, \mathsf{f}_n^i(\overline{s_n}) \twoheadrightarrow \langle t_n, w_n\rangle$$

by a new rule of the form

$$\mathsf{f}^{-1}(r, \beta(\overline{y}, \overline{w_n})) \to \langle\overline{s_0}\rangle \Leftarrow \mathsf{f}_n^{-1}(t_n, w_n) \twoheadrightarrow \langle\overline{s_n}\rangle, \ldots, \mathsf{f}_1^{-1}(t_1, w_1) \twoheadrightarrow \langle\overline{s_1}\rangle$$

Injectivization & Inversion: An example

$$\begin{array}{rrcl} \beta_1 : & \mathsf{add}(0, y) & \to & y \\ \beta_2 : & \mathsf{add}(\mathsf{s}(x), y) & \to & \mathsf{s}(x_1) \Leftarrow \mathsf{add}(x, y) \twoheadrightarrow x_1 \\ \beta_3 : & \mathsf{fst}(x, y) & \to & x \end{array}$$

$$\begin{array}{rcl} \mathsf{add}^i(0, y) & \to & \langle y, \beta_1 \rangle \\ \mathsf{add}^i(\mathsf{s}(x), y) & \to & \langle \mathsf{s}(x_1), \beta_2(w_1) \rangle \Leftarrow \mathsf{add}^i(x, y) \twoheadrightarrow \langle x_1, w_1 \rangle \\ \mathsf{fst}^i(x, y) & \to & \langle x, \beta_3(y) \rangle \end{array}$$

$$\begin{array}{rcl} \mathsf{add}^{-1}(y, \beta_1) & \to & \langle 0, y \rangle \\ \mathsf{add}^{-1}(\mathsf{s}(x_1), \beta_2(w_1)) & \to & \langle \mathsf{s}(x), y \rangle \Leftarrow \mathsf{add}^{-1}(x_1, w_1) \twoheadrightarrow \langle x, y \rangle \\ \mathsf{fst}^{-1}(x, \beta_3(y)) & \to & \langle x, y \rangle \end{array}$$

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\mathsf{fst}^i(x, y) & \rightarrow & \langle x, \beta_3(y) \rangle
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\mathsf{fst}^{-1}(x, \beta_3(y)) & \rightarrow & \langle x, y \rangle
\end{array}$$

Conclusions

Conclusions and future work

In summary,

- we have introduced a reversible extension of term rewriting
- we have defined two transformations so that standard rewriting becomes reversible
- we have successfully applied it to some problems

Future work: Look for conditions that allow us to remove from trace terms

- variable bindings
- rule labels
- complete traces

Thanks for your attention!

$\sim$15:30h *Reversible Term Rewriting in Practice*
WPTE'16 Naoki Nishida, Adrián Palacios, Germán Vidal